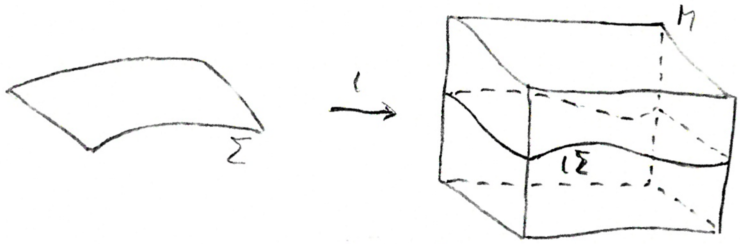
Submanifolds - tangent vectors and normal forms

embedding Σ in M

$\iota : \Sigma \to M$

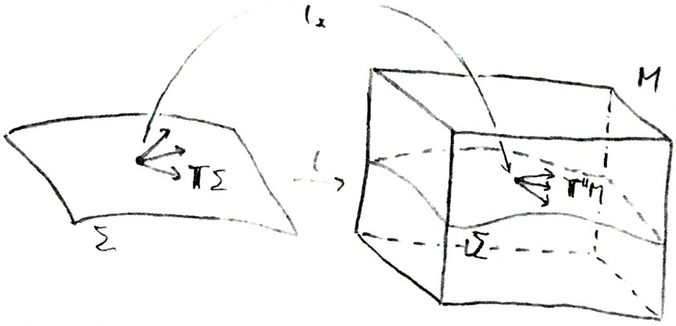


$\mathbb{T}\Sigma$ tangent bundle

$\mathbb{T}^{\shortparallel}M \subset \mathbb{T}M$ tangent vectors to Σ embedded to $\mathbb{T}M$

$\iota_* : \mathbb{T}\Sigma \to \mathbb{T}^{\shortparallel}M \subset \mathbb{T}M$

$a \to \iota_* a$



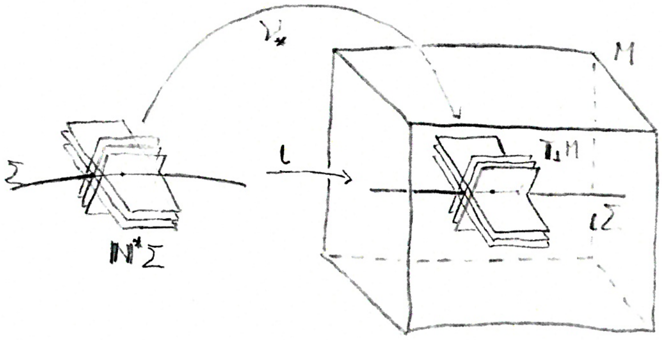
$\mathbb{N}^*\Sigma$ normal covector bundle

$\mathbb{T}_\perp M \subset \mathbb{T}^*M$ normal forms embedded to $\mathbb{T}^*M$

$\mathbb{N}^*\Sigma$ abstract equivalent of $\mathbb{T}_\perp M$

$\nu_* : \mathbb{N}^*\Sigma \to \mathbb{T}_\perp M \subset \mathbb{T}^*M$

$K \to \nu_* K$

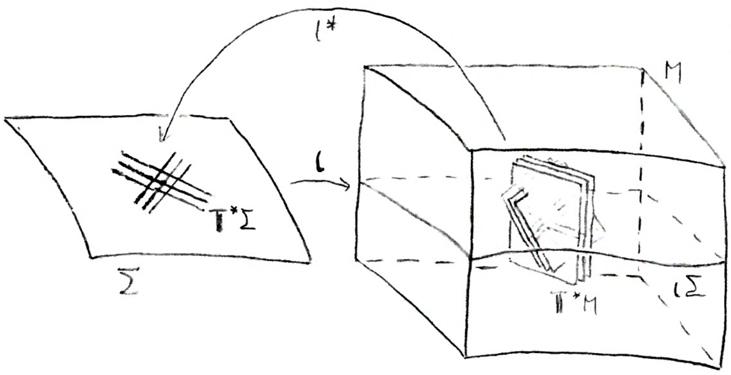


$\mathbb{T}^*\Sigma$ cotangent bundle

$\iota^* : \mathbb{T}^*M \to \mathbb{T}^*\Sigma$

$\omega \to \omega|_{\mathbb{T}\Sigma} = \iota^*\omega$

$\mathbb{T}^*\Sigma$ is not embedded in $\mathbb{T}^*M$

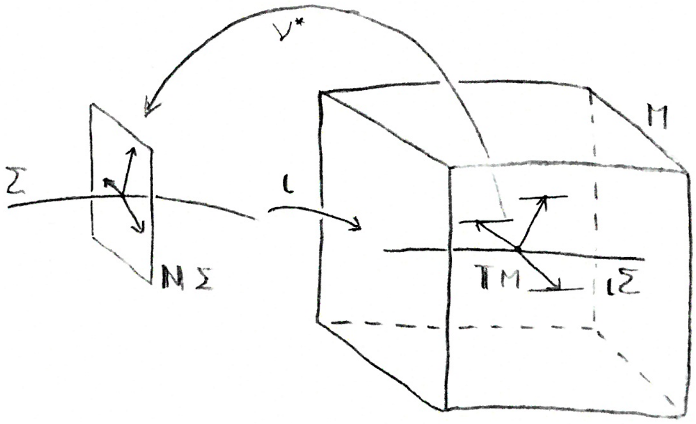


$\mathbb{N}\Sigma$ normal vector bundle

$\nu^* : \mathbb{T}M \to \mathbb{N}\Sigma$

$u \to u|_{\mathbb{N}\Sigma} = \nu^* u$

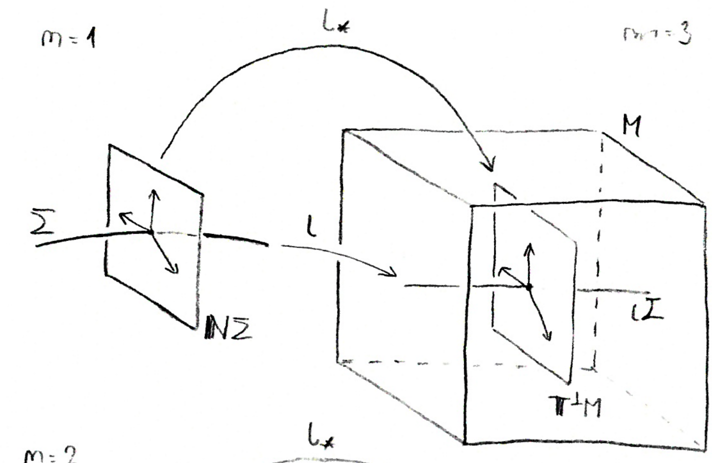
$\mathbb{N}\Sigma$ is not embedded in $\mathbb{T}M$

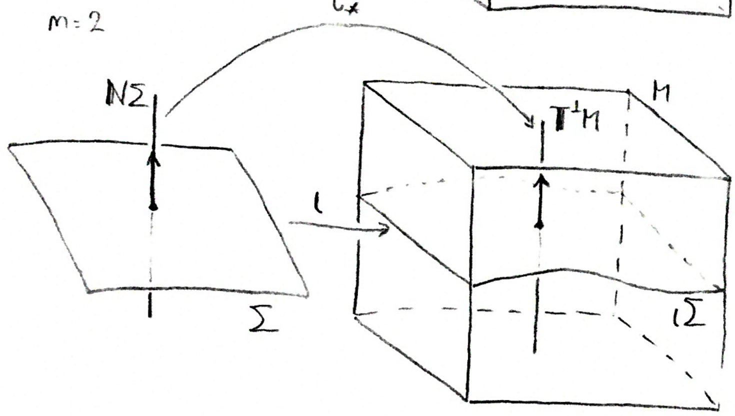


$\mathbb{T}^\perp M$ normal vectors embedded in $\mathbb{T}M$

$\iota_* : \mathbb{N}\Sigma \to \mathbb{T}^\perp M \subset \mathbb{T}M$

defined using ${}^\perp\delta$

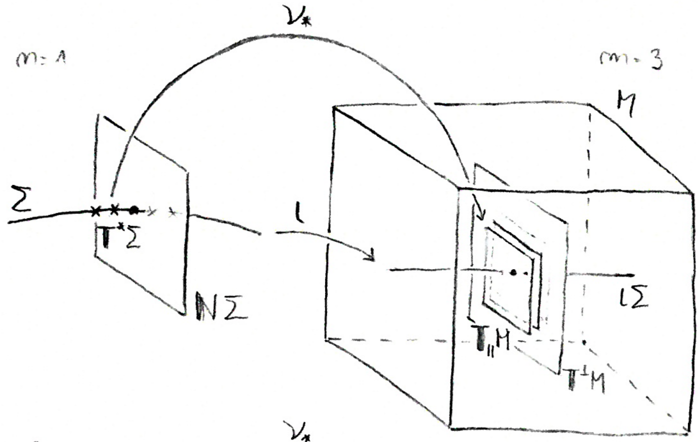


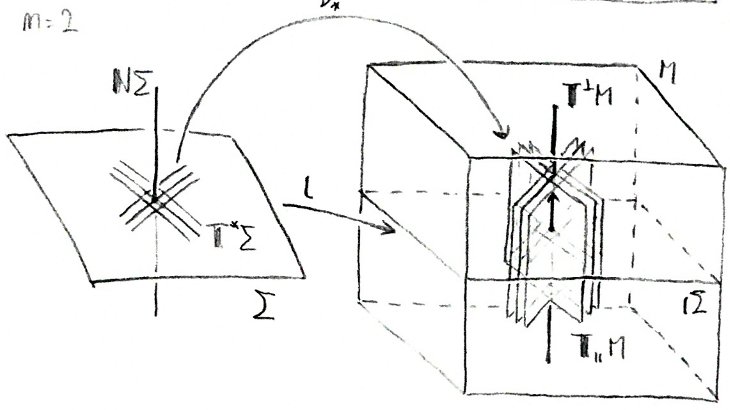


$\mathbb{T}_\parallel M$ tangent covectors embedded in $\mathbb{T}^* M$

$\nu_* : \mathbb{T}^*\Sigma \to \mathbb{T}_\parallel M \subset \mathbb{T}^* M$

defined using ${}^\parallel\delta$





Submanifold with projector

$\iota : \Sigma \to M$ submanifold Σ in M

natural structures

$\iota_* : \mathbb{T}\Sigma \to \mathbb{T}^{\parallel}M \subset \mathbb{T}M$ — embedding of tangent vectors

$\iota^* : \mathbb{T}^*M \to \mathbb{T}^*\Sigma$ — restriction of covectors on tang. covect.

$\nu_* : \mathbb{N}^*\Sigma \to \mathbb{T}_{\perp}M \subset \mathbb{T}^*M$ — embedding of normal covectors

$\nu^* : \mathbb{T}M \to \mathbb{N}\Sigma$ — restriction of vectors on normal vect.

projectors on tangent and normal directions

$\mathbb{T}^{\perp}M$ choice of normal vector subspace $\quad \mathbb{T}^{\perp}M \subset \mathbb{T}M \quad \mathbb{T}M = \mathbb{T}^{\parallel}M \oplus \mathbb{T}^{\perp}M$

$\mathbb{T}_{\parallel}M$ choice of tangent covector subspace $\quad \mathbb{T}_{\parallel}M \subset \mathbb{T}^*M \quad \mathbb{T}^*M = \mathbb{T}_{\parallel}M \oplus \mathbb{T}_{\perp}M$

duality $\mathbb{T}^{\perp}M \leftrightarrow \mathbb{T}_{\parallel}M$ — one space determines other

$k \in \mathbb{T}^{\perp}M \iff \forall \alpha \in \mathbb{T}_{\parallel}M \quad \alpha \cdot k = 0$

$\alpha \in \mathbb{T}_{\parallel}M \iff \forall k \in \mathbb{T}^{\perp}M \quad \alpha \cdot k = 0$

projectors

$^{\parallel}\delta : \mathbb{T}M \to \mathbb{T}^{\parallel}M \quad \ker {}^{\parallel}\delta = \mathbb{T}^{\perp}M \quad \operatorname{img} {}^{\parallel}\delta = \mathbb{T}^{\parallel}M \quad {}^{\parallel}\delta + {}^{\perp}\delta = \delta$

$^{\perp}\delta : \mathbb{T}M \to \mathbb{T}^{\perp}M \quad \ker {}^{\perp}\delta = \mathbb{T}^{\parallel}M \quad \operatorname{img} {}^{\perp}\delta = \mathbb{T}^{\perp}M \quad {}^{\parallel}\delta\, {}^{\perp}\delta = {}^{\perp}\delta\, {}^{\parallel}\delta = 0$

$^{\parallel}\delta : \mathbb{T}^*M \to \mathbb{T}_{\parallel}M \quad \ker {}^{\parallel}\delta = \mathbb{T}_{\perp}M \quad \operatorname{img} {}^{\parallel}\delta = \mathbb{T}_{\parallel}M$

$^{\perp}\delta : \mathbb{T}M \to \mathbb{T}_{\perp}M \quad \ker {}^{\perp}\delta = \mathbb{T}_{\parallel}M \quad \operatorname{img} {}^{\perp}\delta = \mathbb{T}_{\perp}M$

choice $\mathbb{T}^{\perp}M, \mathbb{T}_{\parallel}M$ allows extension of isomorphisms ι_* and ν_*

$\iota_* : \mathbb{T}\Sigma \to \mathbb{T}^{\parallel}M \subset \mathbb{T}M \quad \iota_* a \in \mathbb{T}^{\parallel}M \quad \nu^*\iota_* a = 0$

$\iota_* : \mathbb{N}\Sigma \to \mathbb{T}^{\perp}M \subset \mathbb{T}M \quad \iota_* k \in \mathbb{T}^{\perp}M \quad \nu^*\iota_* k = k$ — extension on $\mathbb{N}\Sigma$

$\iota_* : \mathbb{T}\oplus\mathbb{N}\Sigma \to \mathbb{T}M$ — vector isomorphism

$\nu_* : \mathbb{N}^*\Sigma \to \mathbb{T}_{\perp}M \subset \mathbb{T}M \quad \nu_* k \in \mathbb{T}_{\perp}M \quad \iota^*\nu_* k = 0$

$\nu_* : \mathbb{T}^*\Sigma \to \mathbb{T}_{\parallel}M \subset \mathbb{T}M \quad \nu_* \alpha \in \mathbb{T}_{\parallel}M \quad \iota^*\nu_* \alpha = \alpha$ — extension on $\mathbb{T}^*\Sigma$

$\nu_* : \mathbb{T}^*\oplus\mathbb{N}^*\Sigma \to \mathbb{T}^*M$ — covector isomorphism

$\iota\nu_* : (\mathbb{T}+\mathbb{N})^p_q\Sigma \to \mathbb{T}^p_q M$ — isomorphism induced by ι_* on vect. and by ν_* on covect.

if projectors $^{\parallel}\delta, {}^{\perp}\delta$ or subspaces $\mathbb{T}^{\perp}M$, $\mathbb{T}_{\parallel}M$ are fixed
it is not necessary to distinguish spaces

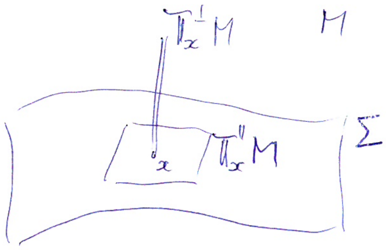
$(\mathbb{T}+\mathbb{N})^p_q\Sigma \quad a \quad \mathbb{T}^p_q M$

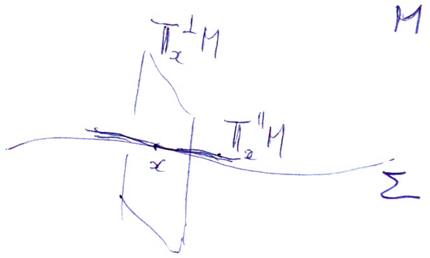
Submanifold of (pseudo)riemannian manifold

$\iota : \Sigma \hookrightarrow M$

$\dim M = m$

$\dim N = n$





M with a metric g

induces an ortogonal splitting

$T_x M = T_x^\parallel M + T_x^\perp M$

$T_x^\parallel M = T_x \Sigma$ tangent vectors

$T_x^\perp M$ normal vectors

$k \in T_x^\perp M \iff \forall a \in T_x^\parallel M \quad k \cdot g \cdot a = 0$

induces an embedding of tangent covectors $T^*\Sigma$ into T^*M

$T_x^* M = T_{x\parallel} M + T_{x\perp} M$

$T_{x\perp} M = N_x \Sigma$ normal covectors

$T_{x\parallel} M$ tangent covectors

$\alpha \in T_{x\parallel} M \iff \forall k \in T_x^\perp M \quad \alpha \cdot k = 0$

$\iff \forall \kappa \in T_{x\perp} M \quad \alpha \cdot g^{-1} \cdot \kappa = 0$

metric splits into orthogonal components

$g = {}^\parallel g + {}^\perp g \qquad g^{-1} = {}^\parallel g^{-1} + {}^\perp g^{-1} \qquad \delta = {}^\parallel\delta + {}^\perp\delta \qquad {}^\parallel\delta, {}^\perp\delta$ projectors

consistency of lowering and raising of indices

$a = {}^\sharp\alpha \in T^\parallel M \iff \alpha = {}^\flat a \in T_\parallel M$

$k = {}^\sharp\kappa \in T^\perp M \iff \kappa = {}^\flat k \in T_\perp M$

$\Uparrow$ there is no mixed component $a \cdot g \cdot k = 0 \quad a \in T^\parallel M \quad k \in T^\perp M$

Notation

$\delta = {}^{\parallel}\delta + {}^{\perp}\delta$

${}^{\parallel}\delta$ projector onto $T^{\parallel}M$

${}^{\perp}\delta$ projector onto $T^{\perp}M$

${}^{\parallel}A^{a\ldots}_{b\ldots} = {}^{\parallel}\delta^{a}_{k}\cdots{}^{\parallel}\delta^{l}_{b}\cdots A^{k\ldots}_{l\ldots}$ projection onto $T^{\parallel}_{\parallel}M$ in all indices

${}^{\perp}A^{a\ldots}_{b\ldots} = {}^{\perp}\delta^{a}_{k}\cdots{}^{\perp}\delta^{l}_{b}\cdots A^{k\ldots}_{l\ldots}$ projection onto $T^{\perp}_{\perp}M$ in all indices

$A^{\ldots\parallel a\ldots}_{\ldots\perp b\ldots}$ ci $A^{\ldots\parallel\ldots}_{\ldots\perp\ldots}$ mixed projection

$$A^{\ldots\parallel a\ldots}_{\ldots\perp b\ldots} = {}^{\parallel}\delta^{a}_{k}\,{}^{\perp}\delta^{l}_{b}\,A^{\ldots k\ldots}_{\ldots l\ldots}$$

${}^{\parallel}A$ resp. ${}^{\perp}A$ indicate that ${}^{\parallel}A$ is tangent, resp. ${}^{\perp}A$ is normal in all indices

we will use

$a, b, c, \ldots \in T^{\parallel}M$

$\alpha, \beta, \delta, \ldots \in T_{\parallel}M$

$k, l, m, \ldots \in T^{\perp}M$

$\kappa, \lambda, \mu, \ldots \in T_{\perp}M$

metric

${}^{\parallel}g = g_{\parallel\parallel}$ metric on $T\Sigma \equiv T^{\parallel}M$

${}^{\perp}g = g_{\perp\perp}$ metric on normal vectors $N\Sigma \equiv T^{\perp}M$

${}^{\perp}g^{-1} = g^{-1\,\perp\perp}$ metric on normal covectors $N^{*}\Sigma \equiv T_{\perp}M$

$g_{\perp\parallel} = g_{\parallel\perp} = 0$ orthogonality

Splitting of covariant derivative on $\mathbb{T}M$

∇ general covariant derivative on $\mathbb{T}M$

action on tangent bundle $\mathbb{T}\Sigma$

$\nabla_c a = \overset{\parallel}{\nabla}_c a + \mathrm{II}_c \cdot a$ Gauss formula

$\overset{\parallel}{\nabla}_c a = {}^{\parallel}(\nabla_c a)$ covariant der. on $\mathbb{T}\Sigma$

$\mathrm{II}_c \cdot a = c \cdot \mathrm{II} \cdot a = {}^{\perp}(\nabla_c a)$ second fundamental form

$\mathrm{II}_c = \mathrm{II}_{c\,\parallel}^{\perp}$ $\qquad \mathrm{II}_c \in \mathbb{T}^{\perp}_{\parallel} M \leftrightarrow \mathbb{T}^{*} \otimes \mathbb{N}\Sigma$

$\mathrm{II} = \mathrm{II}^{\perp}_{\parallel\,\parallel}$ $\qquad \mathrm{II} \in \mathbb{T}^{\perp}_{\parallel\parallel} M \leftrightarrow \mathbb{T}^{0}_{2} \otimes \mathbb{N}\Sigma$

proof: ${}^{\parallel}(\nabla_c a)$ satisfies all properties of a covariant derivative
${}^{\perp}(\nabla_c a)$ ultralocality in a: ${}^{\perp}(\nabla_c (fa)) = {}^{\perp}(f\nabla_c a + c[f]\,a) = f\,{}^{\perp}(\nabla_c a)$

extension of $\overset{\parallel}{\nabla}$ on tangent tensors $\quad \mathbb{T}^{\parallel p}_{\parallel q} M \leftrightarrow \mathbb{T}^{p}_{q}\Sigma$

$\overset{\parallel}{\nabla}_c A = {}^{\parallel}(\nabla_c A)$ for $A = {}^{\parallel}A$

proof: a standard extension on covectors:

$(\overset{\parallel}{\nabla}_c \alpha) \cdot a = c[\alpha \cdot a] - \alpha \cdot \overset{\parallel}{\nabla}_c a = \nabla_c(\alpha \cdot a) - \alpha \cdot \nabla_c a = (\nabla_c \alpha) \cdot a = {}^{\parallel}(\nabla_c \alpha) \cdot a$

$\Rightarrow \overset{\parallel}{\nabla}_c \alpha = {}^{\parallel}(\nabla_c \alpha)$

commutation of projection with tens. product + Leibniz rule $\Rightarrow$ extension on tensors

it naturally holds

$\overset{\parallel}{\nabla}_c {}^{\parallel}\delta = 0$

proof: $\overset{\parallel}{\nabla}_c a = \overset{\parallel}{\nabla}_c({}^{\parallel}\delta \cdot a) = {}^{\parallel}(\nabla_c({}^{\parallel}\delta \cdot a)) = {}^{\parallel}((\nabla_c {}^{\parallel}\delta) \cdot a + {}^{\parallel}\delta \cdot \nabla_c a) = {}^{\parallel}(\nabla_c {}^{\parallel}\delta) \cdot a + {}^{\parallel}(\nabla_c a) = (\overset{\parallel}{\nabla}_c {}^{\parallel}\delta) \cdot a + \overset{\parallel}{\nabla}_c a$

splitting of torsion

$T^{\parallel}_{\parallel\parallel} = \overset{\parallel}{T}$

$T^{\perp}_{\parallel\parallel} = \mathrm{II} - \mathrm{II}^{T}$

proof:

$T^{\parallel}_{a,b} = {}^{\parallel}(\nabla_a b - \nabla_b a - [a,b]) = \overset{\parallel}{\nabla}_a b - \overset{\parallel}{\nabla}_b a - [a,b] = \overset{\parallel}{T}_{a,b}$

$T^{\perp}_{a,b} = {}^{\perp}(\nabla_a b - \nabla_b a - [a,b]) = a \cdot \mathrm{II} \cdot b - b \cdot \mathrm{II} \cdot a$

action on normal bundle $\mathbb{N}\Sigma$

$$\nabla_c k = \overset{\perp}{\nabla}_c k - \overline{\mathrm{II}}_c \cdot k \qquad \text{Weingarten formula}$$

$$\overset{\perp}{\nabla}_c k = {}^{\perp}(\nabla_c k) \qquad \text{covariant derivative on } \mathbb{N}\Sigma$$

$$\overline{\mathrm{II}}_c \cdot k = -{}^{\parallel}(\nabla_c k) \qquad \text{shape operator } (S_k \cdot c \equiv \overline{\mathrm{II}}_c k)$$

$$\overline{\mathrm{II}}_c = \overline{\mathrm{II}}_{c\perp}{}^{\parallel} \qquad \mathrm{II}_c \in \mathbb{T}^{\parallel}_{\perp} M \leftrightarrow \mathbb{T} \otimes \mathbb{N}^* \Sigma$$

proof: ${}^{\perp}(\nabla_c k)$ satisfies all properties of covariant der. on $\mathbb{N}\Sigma$

${}^{\parallel}(\nabla_c k)$ ultralocality in k $\quad {}^{\parallel}(\nabla_c(fk)) = {}^{\parallel}(f \nabla_c k + c[f] k) = f\, {}^{\parallel}(\nabla_c k)$

extension of $\overset{\perp}{\nabla}$ on normal tensors $\quad \mathbb{T}^{\perp p}_{\perp q} M \leftrightarrow \mathbb{N}^p_q \Sigma$

$$\overset{\perp}{\nabla}_c M = {}^{\perp}(\nabla_c M) \qquad \text{for} \quad M = {}^{\perp}M$$

proof: a standard extension on covectors:

$(\overset{\perp}{\nabla}_c \mu) \cdot k = c[\mu \cdot k] - \mu \cdot \overset{\perp}{\nabla}_c k = \nabla_c(\mu \cdot k) - \mu \cdot \nabla_c k = (\nabla_c \mu) \cdot k = {}^{\perp}(\nabla_c \mu) \cdot k$

$\Rightarrow \overset{\perp}{\nabla}_c \mu = {}^{\perp}(\nabla_c \mu)$

commutation with tensor product + Leibniz rule $\Rightarrow$ extension on tensors

it naturally holds:

$$\overset{\perp}{\nabla}_c {}^{\perp}\delta = 0$$

restriction of ∇ on adjusted covariant der. $\bar{\bar{\nabla}}$

$\tilde{\nabla} = \overset{\parallel}{\nabla} \oplus \overset{\perp}{\nabla}$ на $\mathbb{T}M \leftarrow (\mathbb{T} \oplus \mathbb{N})\Sigma$

$\bar{\bar{\nabla}} = \overset{\parallel}{\nabla}$ на $\mathbb{T}^{\parallel}M \leftrightarrow \mathbb{T}\Sigma$

$\bar{\bar{\nabla}} = \overset{\perp}{\nabla}$ на $\mathbb{T}^{\perp}M \leftrightarrow \mathbb{N}\Sigma$

natural extension to $\mathbb{T}^p_q M \leftrightarrow (\mathbb{T} \oplus \mathbb{N})^p_q \Sigma$

$\bar{\nabla}\,{}^{\parallel}\delta = 0 \qquad \bar{\nabla}\,{}^{\perp}\delta = 0 \qquad \Rightarrow \bar{\nabla}$ adjusted cov. der.

relation ∇ and $\bar{\bar{\nabla}}$

$\nabla = \bar{\bar{\nabla}} + \mathbb{H}$ $\qquad \mathbb{H}$ generated by H

$H = \mathrm{I\!I} - \bar{\bar{\mathrm{I\!I}}} \qquad H^{\perp}{}_{\parallel} = \mathrm{I\!I} \qquad H^{\parallel}{}_{\perp} = -\bar{\bar{\mathrm{I\!I}}}$

$\Downarrow$

$\nabla_c a = \overset{\parallel}{\nabla}_c a + \mathrm{I\!I}_c \cdot a \qquad \nabla_c k = \overset{\perp}{\nabla}_c k - \bar{\bar{\mathrm{I\!I}}}_c \cdot k$

$\nabla_c \alpha = \overset{\parallel}{\nabla}_c \alpha + \alpha \cdot \bar{\bar{\mathrm{I\!I}}}_c \qquad \nabla_c \kappa = \overset{\perp}{\nabla}_c \kappa - \kappa \cdot \mathrm{I\!I}_c$

derivatives of projectors

$\nabla_c{}^{\parallel}\delta = \mathrm{I\!I}_c + \bar{\bar{\mathrm{I\!I}}}_c \qquad \Leftarrow \nabla_c{}^{\parallel}\delta = \bar{\bar{\nabla}}_c{}^{\parallel}\delta + \mathrm{I\!I}_c \cdot {}^{\parallel}\delta + {}^{\parallel}\delta \cdot \bar{\bar{\mathrm{I\!I}}}_c = \mathrm{I\!I}_c + \bar{\bar{\mathrm{I\!I}}}_c$

$\nabla_c{}^{\perp}\delta = -(\mathrm{I\!I}_c + \bar{\bar{\mathrm{I\!I}}}_c) \qquad \Leftarrow \nabla_c{}^{\perp}\delta = \bar{\bar{\nabla}}_c{}^{\perp}\delta - \bar{\bar{\mathrm{I\!I}}}_c \cdot {}^{\perp}\delta - {}^{\perp}\delta \cdot \mathrm{I\!I}_c = -(\mathrm{I\!I}_c + \bar{\bar{\mathrm{I\!I}}}_c)$

curvature on $(\mathbb{T} + \mathbb{N})\Sigma$

$\bar{R} = \overset{\parallel}{R} + \overset{\perp}{R}$

$\overset{\parallel}{R} = \overset{\parallel}{R}_{\parallel\parallel}{}^{\parallel}{}_{\parallel} \qquad \overset{\perp}{R} = \overset{\perp}{R}_{\parallel\parallel}{}^{\perp}{}_{\perp} \qquad \bar{R} = \bar{R}_{\parallel\parallel}$

Orthogonal splitting and metric covariant derivative

$g = {}^{\perp}g + {}^{\parallel}g$ tj. $g_{\perp\parallel} = g_{\parallel\perp} = 0$ $\quad \mathbb{T}^{\parallel}M \perp \mathbb{T}^{\perp}M$

$\nabla g = 0$

$\Downarrow$

$\overset{\parallel}{\nabla}$ metric on $\mathbb{T}\Sigma$ $\quad \overset{\parallel}{\nabla}{}^{\parallel}g = 0$

$\overset{\perp}{\nabla}$ metric on $\mathbb{N}\Sigma$ $\quad \overset{\perp}{\nabla}{}^{\perp}g = 0$

$\bar{\nabla}$ metric on $\mathbb{T}M \leftrightarrow (\mathbb{T}\oplus\mathbb{N})\Sigma$ $\quad \bar{\nabla}g = 0$

$g\cdot H_c + (g\cdot H_c)^T = 0$ tj. $H_{\underline{a}mn} = -H_{\underline{a}nm}$ $\quad H_{\underline{a}mn} = g_{\underline{a}k} H^{k}{}_{\underline{a}\,mn}$

${}^{\perp}g\cdot \mathbb{I} = \bar{\mathbb{I}}\cdot{}^{\parallel}g$ tj. $\mathbb{I}_{\parallel\perp\parallel}^{\downarrow} = \bar{\mathbb{I}}_{\parallel\perp\parallel}^{\downarrow}$

proof:

$\overset{\parallel}{\nabla}_c {}^{\parallel}g = {}^{\parallel}(\nabla_c {}^{\parallel}g) = {}^{\parallel}(\nabla_c({}^{\parallel}\delta\cdot g)) = {}^{\parallel}((\nabla_c{}^{\parallel}\delta)\cdot g) = {}^{\parallel}(\nabla_c{}^{\parallel}\delta)\cdot{}^{\parallel}g = (\overset{\parallel}{\nabla}_c{}^{\parallel}\delta)\cdot{}^{\parallel}g = 0$

$\overset{\perp}{\nabla}_c {}^{\perp}g = {}^{\perp}(\nabla_c {}^{\perp}g) = {}^{\perp}(\nabla_c({}^{\perp}\delta\cdot g)) = {}^{\perp}((\nabla_c{}^{\perp}\delta)\cdot g) = {}^{\perp}(\nabla_c{}^{\perp}\delta)\cdot{}^{\perp}g = (\overset{\perp}{\nabla}_c{}^{\perp}\delta)\cdot{}^{\perp}g = 0$

$\bar{\nabla}_c g = \overset{\parallel}{\nabla}_c{}^{\parallel}g + \overset{\perp}{\nabla}_c{}^{\perp}g = 0$

$0 = \nabla_c g = \bar{\nabla}_c g - g\cdot H_c - (g\cdot H_c)^T = -(g\cdot H_c + (g\cdot H_c)^T)$

$(g\cdot H_c + (g\cdot H_c)^T)_{\perp\parallel} = {}^{\perp}g\cdot H_c{}^{\perp}{}_{\parallel} + ({}^{\parallel}g\cdot H_c{}^{\parallel}{}_{\perp})^T = {}^{\perp}g\,\mathbb{I}_c - \bar{\mathbb{I}}_c\cdot{}^{\parallel}g = 0$

derivatives of metrics

$\nabla_c {}^{\parallel}g_{ab} = \mathbb{I}_{cab} + \mathbb{I}_{cba}$ $\qquad \nabla_{\parallel c} {}^{\parallel}g_{ab} = \mathbb{I}_{cab} + \mathbb{I}_{cba}$

$\nabla_c {}^{\perp}g_{ab} = -\bar{\mathbb{I}}_{cab} - \bar{\mathbb{I}}_{cba}$ $\qquad \nabla_{\parallel c} {}^{\perp}g_{ab} = -\bar{\mathbb{I}}_{cab} - \bar{\mathbb{I}}_{cba}$

$\nabla_c {}^{\parallel}g^{ab} = \mathbb{I}_c{}^{ab} + \bar{\mathbb{I}}_c{}^{ba}$ $\qquad \nabla_{\parallel c} {}^{\parallel}g^{ab} = \mathbb{I}_c{}^{ab} + \bar{\mathbb{I}}_c{}^{ba}$

$\nabla_c {}^{\perp}g^{ab} = -\bar{\mathbb{I}}_c{}^{ab} - \mathbb{I}_c{}^{ba}$ $\qquad \nabla_{\parallel c} {}^{\perp}g^{ab} = -\bar{\mathbb{I}}_c{}^{ab} - \bar{\mathbb{I}}_c{}^{ba}$

proof:

$\nabla_{\parallel c} {}^{\parallel}g_{ab} = \bar{\nabla}_{\parallel c} {}^{\parallel}g_{ab} + \mathbb{H}_c {}^{\parallel}g_{ab} = \mathbb{I}_{ca}{}^{k}\,{}^{\parallel}g_{kb} + \mathbb{I}_{cb}{}^{k}\,{}^{\parallel}g_{ak} = \mathbb{I}_{cab} + \mathbb{I}_{cba}$

$\nabla_{\parallel c} {}^{\perp}g_{ab} = \bar{\nabla}_{\parallel c} {}^{\perp}g_{ab} + \mathbb{H}_c {}^{\perp}g_{ab} = -\bar{\mathbb{I}}_{ca}{}^{k}\,{}^{\perp}g_{kb} - \bar{\mathbb{I}}_{cb}{}^{k}\,{}^{\perp}g_{ak} = -\bar{\mathbb{I}}_{cba} - \bar{\mathbb{I}}_{cab}$

obdobně ${}^{\parallel}g^{ab}$ a ${}^{\perp}g^{ab}$

Splitting of curvature

$\nabla_{\parallel} = \bar{\nabla} + \mathbb{H}$ understood as a covariant der. on Σ

$H_c{}^{\perp}{}_{\parallel} = \mathrm{II}_c \qquad H_c{}^{\parallel}{}_{\perp} = -\bar{\mathrm{II}} \qquad H_c = \mathrm{II}_c - \bar{\mathrm{II}}_c \qquad \bar{\nabla}^{\perp}\delta = \bar{\nabla}^{\parallel}\delta = 0$

$$R_{\parallel\parallel} = \bar{R} + {}^{\bar{\nabla}^{\parallel}}d H + H \wedge H$$

$$R_{\parallel a \parallel b}{}^{m}{}_{n} = \bar{R}_{ab}{}^{m}{}_{n} + ({}^{\bar{\nabla}^{\parallel}}d_a H_b)^{m}{}_{n} + (H_a \wedge H_b)^{m}{}_{n}$$

$$= \overset{\parallel}{R}_{ab}{}^{m}{}_{n} + \overset{\perp}{R}_{ab}{}^{m}{}_{n} + \bar{\nabla}_a H_b{}^{m}{}_{n} - \bar{\nabla}_b H_a{}^{m}{}_{n} + \mathbb{T}^{c}_{ab} H_c{}^{m}{}_{n} + H_a{}^{m}{}_{k} H_b{}^{k}{}_{n} - H_b{}^{m}{}_{k} H_a{}^{k}{}_{n}$$

$\Downarrow$

$$R_{\parallel a \parallel b}{}^{\parallel m}{}_{\parallel n} = \overset{\parallel}{R}_{ab}{}^{m}{}_{n} - \bar{\mathrm{II}}_{ak}{}^{m} \mathrm{II}_{b}{}^{k}{}_{n} + \bar{\mathrm{II}}_{bk}{}^{m} \mathrm{II}_{a}{}^{k}{}_{n}$$ Gauss equation

$$R_{\parallel a \parallel b}{}^{\perp m}{}_{\perp n} = \overset{\perp}{R}_{ab}{}^{m}{}_{n} - \mathrm{II}_{a}{}^{m}{}_{k} \bar{\mathrm{II}}_{bn}{}^{k} + \mathrm{II}_{b}{}^{m}{}_{k} \bar{\mathrm{II}}_{an}{}^{k}$$ Ricci(-Kühne) eq.

$$R_{\parallel a \parallel b}{}^{\perp m}{}_{\parallel n} = \bar{\nabla}_a \mathrm{II}_{b}{}^{m}{}_{n} - \bar{\nabla}_b \mathrm{II}_{a}{}^{m}{}_{n} + \mathbb{T}^{c}_{ab} \mathrm{II}_{c}{}^{m}{}_{n} = ({}^{\bar{\nabla}^{\parallel}}d_a \mathrm{II}_b)^{m}{}_{n}$$ Codazzi-

$$R_{\parallel a \parallel b}{}^{\parallel m}{}_{\perp n} = -\bar{\nabla}_a \bar{\mathrm{II}}_{bn}{}^{m} + \bar{\nabla}_b \bar{\mathrm{II}}_{an}{}^{m} - \mathbb{T}^{c}_{ab} \bar{\mathrm{II}}_{cn}{}^{m} = -({}^{\bar{\nabla}^{\parallel}}d_a \bar{\mathrm{II}}_b)_{n}{}^{m}$$ Mainardi eq.

metric derivative

$$\mathrm{II}_{a\overset{\downarrow}{m}n} = \bar{\mathrm{II}}_{am\overset{\downarrow}{n}}$$

$$R_{\parallel a \parallel b \parallel c \parallel d} = \overset{\parallel}{R}_{abcd} - (\mathrm{II}_{a}{}^{m}{}_{c} \mathrm{II}_{b}{}^{n}{}_{d} - \mathrm{II}_{b}{}^{m}{}_{c} \mathrm{II}_{a}{}^{n}{}_{d})\, {}^{\perp}g_{mn}$$

$$R_{\parallel a \parallel b}{}^{\perp m \perp n} = \overset{\perp}{R}_{ab}{}^{mn} - (\mathrm{II}_{a}{}^{m}{}_{c} \mathrm{II}_{b}{}^{n}{}_{d} - \mathrm{II}_{b}{}^{m}{}_{c} \mathrm{II}_{a}{}^{n}{}_{d})\, {}^{\parallel}g^{cd}$$

$$R_{\parallel a \parallel b}{}^{\perp m}{}_{\parallel n} = -R_{\parallel a \parallel b \parallel n}{}^{\perp m} = \bar{\nabla}_a \mathrm{II}_{b}{}^{m}{}_{n} - \bar{\nabla}_b \mathrm{II}_{a}{}^{m}{}_{n} + \mathbb{T}^{c}_{ab} \mathrm{II}_{c}{}^{m}{}_{n} = ({}^{\bar{\nabla}^{\parallel}}d_a \mathrm{II}_b)^{m}{}_{n}$$

Contraction of the second fundamental form and curvature

$$\mathrm{Tr}\,\mathrm{II}_m = \breve{\mathrm{II}}_{cm}{}^{c} \qquad \mathrm{Tr}\,\overline{\mathrm{II}}_m = \mathrm{Tr}\,\mathrm{II}_{\perp m}$$

$$\overline{\mathrm{II}}^2_{ab} = \overline{\mathrm{II}}_{ak}{}^{c}\,\mathrm{II}^{k}{}_{bc} \qquad \overline{\mathrm{II}}^2_{ab} = \mathrm{II}^2_{\parallel a \parallel b}$$

contraction of curvature

$$R_{\parallel c \parallel a}{}^{\parallel c}{}_{\parallel b} = \overset{\parallel}{R}ic_{ab} - \mathrm{Tr}\,\overline{\mathrm{II}}_k\,\mathrm{II}^k_{ab} + \overline{\mathrm{II}}^2_{ab} - \overline{\mathrm{II}}_{ak}{}^{m}\,T^{\perp k}{}_{\parallel b \parallel m} = Ric_{\parallel a \parallel b} - R_{\perp k \parallel a}{}^{\perp k}{}_{\parallel b}$$

$$R_{\parallel a \parallel b}{}^{\parallel c}{}_{\parallel c} = \mathrm{Tr}\,\overset{\parallel}{R}_{ab} - \overline{\mathrm{II}}^2_{ab} + \overline{\mathrm{II}}^2_{ba} = \mathrm{Tr}\,R_{\parallel a \parallel b} - R_{\parallel a \parallel b}{}^{\perp k}{}_{\perp k}$$

$$R_{\parallel a \parallel b}{}^{\perp k}{}_{\perp k} = \mathrm{Tr}\,\overset{\perp}{R}_{ab} + \overline{\mathrm{II}}^2_{ab} - \overline{\mathrm{II}}^2_{ba}$$

$$\mathrm{Tr}\,R_{\parallel a \parallel b} = \mathrm{Tr}\,\overset{\parallel}{R}_{ab} + \mathrm{Tr}\,\overset{\perp}{R}_{ab}$$

$$R_{\parallel c \parallel a}{}^{\parallel c}{}_{\perp n} = \overset{\perp}{\nabla}_a\,\mathrm{Tr}\,\mathrm{II}_n - \overline{\nabla}_c\,\overline{\mathrm{II}}_{an}{}^{c} + \overset{\parallel}{T}{}^{c}{}_{ab}\,\overline{\mathrm{II}}_{cn}{}^{b} = Ric_{\parallel a \perp n} - R_{\perp k \parallel a}{}^{\perp k}{}_{\perp n}$$

metric derivative

$$\mathrm{II}_{a\overset{\downarrow}{k}b} = \breve{\mathrm{II}}_{ak\overset{\downarrow}{b}}$$

$$\mathrm{Tr}\,\mathrm{II}^k = \mathrm{II}^k_{ab}\,{}^{\parallel}g^{ab} \qquad m^2 \ae^2 = (\mathrm{Tr}\,\mathrm{II})^2 = {}^{\perp}g^{k\ell}\,\mathrm{Tr}\,\mathrm{II}_k\,\mathrm{Tr}\,\mathrm{II}_\ell$$

$$\mathrm{II}^2_{ab} = \mathrm{II}_a{}^{k}{}_{c}\,\mathrm{II}_b{}^{\ell}{}_{d}\,{}^{\parallel}g^{cd}\,{}^{\perp}g_{k\ell} = \mathrm{II}^2_{ba} \qquad \mathrm{Tr}\,\mathrm{II}^2 = {}^{\parallel}g^{ab}\,\mathrm{II}^2_{ab} = {}^{\perp}g_{k\ell}\,{}^{\parallel}g^{ab}\,{}^{\parallel}g^{cd}\,\mathrm{II}_a{}^{k}{}_{c}\,\mathrm{II}_b{}^{\ell}{}_{d}$$

contraction of curvature - metric der. without torsion

$$R_{\parallel c \parallel a}{}^{\parallel c}{}_{\parallel b} = \overset{\parallel}{R}ic_{ab} - \mathrm{Tr}\,\mathrm{II}_k\,\mathrm{II}^k_{ab} + \mathrm{II}^2_{ab} = Ric_{\parallel a \parallel b} - R_{\perp k \parallel a}{}^{\perp k}{}_{\parallel b}$$

$$R_{\parallel c \parallel a}{}^{\parallel c}{}_{\perp n} = \overset{\perp}{\nabla}_a\,\mathrm{Tr}\,\mathrm{II}_n - \breve{\nabla}_c\,\overline{\mathrm{II}}_{an}{}^{c} = Ric_{\parallel a \perp n} - R_{\perp k \parallel a}{}^{\perp k}{}_{\perp n} = Ric_{\parallel a \perp n} - R_{\perp k \perp n}{}^{\perp k}{}_{\parallel a}$$

$$R_{\parallel a \parallel b}{}^{\parallel a \parallel b} = \overset{\parallel}{R} - (\mathrm{Tr}\,\mathrm{II})^2 + \mathrm{Tr}\,\mathrm{II}^2 = R - 2Ric_{\perp k}{}^{\perp k} + R_{\perp k \perp \ell}{}^{\perp k \perp \ell}$$

Semi-umbilic splitting of general cov. der.

$$\bar{\mathrm{II}}_{ak}{}^{b} = \frac{1}{m}\,\mathrm{Tr}\,\mathrm{II}_k\,{}^{\parallel}\delta_a^b$$

$$\mathrm{II}^2_{ab} = \bar{\mathrm{II}}_{ak}{}^{c}\,\mathrm{II}_{b}{}^{k}{}_{c} = \frac{1}{m}\,\mathrm{Tr}\,\mathrm{II}_k\,\mathrm{II}^k_{ba}$$

curvature

$$R_{\parallel a\parallel b}{}^{\parallel m}{}_{\parallel n} = \overset{\parallel}{R}_{ab}{}^{m}{}_{n} - \frac{1}{m}\,\mathrm{Tr}\,\mathrm{II}_k\left({}^{\parallel}\delta_a^m\,\mathrm{II}^k_{bn} - {}^{\parallel}\delta_b^m\,\mathrm{II}^k_{an}\right) = \overset{\parallel}{R}_{ab}{}^{m}{}_{n} + \mathrm{II}^2_{na}\,{}^{\parallel}\delta_b^m - \mathrm{II}^2_{nb}\,{}^{\parallel}\delta_a^m$$

$$R_{\parallel a\parallel b}{}^{\perp m}{}_{\perp n} = \overset{\perp}{R}_{ab}{}^{m}{}_{n} - \frac{1}{m}\,\mathrm{Tr}\,\mathrm{II}_n\,T^{\perp m}_{\parallel a\parallel b}$$

$$R_{\parallel a\parallel b}{}^{\perp m}{}_{\parallel n} = \bar{\nabla}_a\mathrm{II}_b{}^{m}{}_{n} - \bar{\nabla}_b\mathrm{II}_a{}^{m}{}_{n} + \mathbb{T}^c_{ab}\,\mathrm{II}_c{}^{m}{}_{n} = (\bar{\mathrm{d}}_a\mathrm{II}_b)^m{}_n$$

$$R_{\parallel a\parallel b}{}^{\parallel m}{}_{\perp n} = \frac{1}{m}\left({}^{\parallel}\delta_a^m\,\overset{\perp}{\nabla}_b\,\mathrm{Tr}\,\mathrm{II}_n - {}^{\parallel}\delta_b^m\,\overset{\perp}{\nabla}_a\,\mathrm{Tr}\,\mathrm{II}_n - \mathbb{T}^m_{ab}\,\mathrm{Tr}\,\mathrm{II}_n\right)$$

contraction of curvature

$$R_{\parallel c\parallel a}{}^{\parallel c}{}_{\parallel b} = \overset{\parallel}{\mathrm{Ric}}_{ab} - \frac{m-1}{m}\,\mathrm{Tr}\,\mathrm{II}_k\,\mathrm{II}^k_{ab} = \overset{\parallel}{\mathrm{Ric}}_{ab} - (m-1)\,\mathrm{II}^2_{ba}$$

$$R_{\parallel a\parallel b}{}^{\parallel m}{}_{\parallel m} = \mathrm{Tr}\,\overset{\parallel}{R}_{ab} + \frac{1}{m}\,\mathrm{Tr}\,\mathrm{II}_k\,T^{\perp k}_{\parallel a\parallel b} = \mathrm{Tr}\,\overset{\parallel}{R}_{ab} - \mathrm{II}^2_{ab} + \mathrm{II}^2_{ba}$$

$$R_{\parallel a\parallel b}{}^{\perp m}{}_{\perp m} = \mathrm{Tr}\,\overset{\perp}{R}_{ab} - \frac{1}{m}\,\mathrm{Tr}\,\mathrm{II}_k\,T^{\perp k}_{\parallel a\parallel b} = \mathrm{Tr}\,\overset{\perp}{R}_{ab} + \mathrm{II}^2_{ab} - \mathrm{II}^2_{ba}$$

$$\mathrm{Tr}\,R_{\parallel a\parallel b} = \mathrm{Tr}\,\overset{\parallel}{R}_{ab} + \mathrm{Tr}\,\overset{\perp}{R}_{ab}$$

$$R_{\parallel c\parallel a}{}^{\parallel c}{}_{\perp n} = \frac{m-1}{m}\,\overset{\perp}{\nabla}_a\,\mathrm{Tr}\,\mathrm{II}_n - \frac{1}{m}\,\mathbb{T}^c_{ca}\,\mathrm{Tr}\,\mathrm{II}_n$$

Semi-umbilic splitting of torsion-free cov. der

$$\tilde{\mathrm{II}}_{ak}{}^{b} = \frac{1}{n} \operatorname{Tr}\mathrm{II}_k \, {}^{\parallel}\delta_a^b \qquad T^c_{ab} = 0$$

$$\overset{2}{\mathrm{II}}_{ab} = \tilde{\mathrm{II}}_{ak}{}^{c}\, \mathrm{II}_b{}^{k}{}_{c} = \frac{1}{n}\operatorname{Tr}\mathrm{II}_k\, \mathrm{II}^k_{ab} \qquad \overset{2}{\mathrm{II}}_{ab} = \overset{2}{\mathrm{II}}_{ba} \Leftarrow \mathrm{II}^k_{ab} = \mathrm{II}^k_{ba}$$

curvature

$$R_{\parallel a \parallel b}{}^{\parallel m}{}_{\parallel n} = \overset{\parallel}{R}_{ab}{}^{m}{}_{n} - \frac{1}{n}\operatorname{Tr}\mathrm{II}_k \left({}^{\parallel}\delta^m_a \mathrm{II}^k_{bn} - {}^{\parallel}\delta^m_b \mathrm{II}^k_{an} \right) = \overset{\parallel}{R}_{ab}{}^{m}{}_{n} - {}^{\parallel}\delta^m_a \overset{2}{\mathrm{II}}_{bn} + {}^{\parallel}\delta^m_b \overset{2}{\mathrm{II}}_{an}$$

$$R_{\parallel a \parallel b}{}^{\perp m}{}_{\perp n} = \overset{\perp}{R}_{ab}{}^{m}{}_{n}$$

$$R_{\parallel a \parallel b}{}^{\perp m}{}_{\parallel n} = \tilde{\nabla}_a \mathrm{II}_b{}^{m}{}_{n} - \tilde{\nabla}_b \mathrm{II}_a{}^{m}{}_{n}$$

$$R_{\parallel a \parallel b}{}^{\parallel m}{}_{\perp n} = \frac{1}{n}\left({}^{\parallel}\delta^m_a \overset{\perp}{\nabla}_b \operatorname{Tr}\mathrm{II}_n - {}^{\parallel}\delta^m_b \overset{\perp}{\nabla}_a \operatorname{Tr}\mathrm{II}_n \right)$$

contraction of curvature

$$R_{\parallel c \parallel a}{}^{\parallel c}{}_{\parallel b} = \overset{\parallel}{Ric}_{ab} - \frac{n-1}{n}\operatorname{Tr}\mathrm{II}_k\, \mathrm{II}^k_{ab} = \overset{\parallel}{Ric}_{ab} - (n-1)\overset{2}{\mathrm{II}}_{ab}$$

$$R_{\parallel a \parallel b}{}^{\parallel m}{}_{\parallel m} = \operatorname{Tr}\overset{\parallel}{R}_{ab} \qquad R_{\parallel a \parallel b}{}^{\perp m}{}_{\perp m} = \operatorname{Tr}\overset{\perp}{R}_{ab} \qquad \operatorname{Tr} R_{\parallel a \parallel b} = \operatorname{Tr}\overset{\parallel}{R}_{ab} + \operatorname{Tr}\overset{\perp}{R}_{ab}$$

$$R_{\parallel c \parallel a}{}^{\parallel c}{}_{\perp n} = \frac{n-1}{n}\overset{\perp}{\nabla}_a \operatorname{Tr}\mathrm{II}_n$$

Totally umbilic submanifolds

metric g_{ab}, Levi-Civita der ∇_a $\qquad \nabla g = 0 \qquad T = 0$

$\mathrm{II}_{a\overset{\downarrow}{k}b} \sim \bar{\mathrm{II}}_{ak\overset{\downarrow}{b}} \qquad \mathrm{II}_{ab}{}^{k} = \mathrm{II}_{ba}{}^{k} \qquad \mathrm{II}_{ab}^{?} = \mathrm{II}_{ba}^{?}$

umbilic $\qquad \bar{\mathrm{II}}_{ak}{}^{b} = \frac{1}{m}\,\mathrm{Tr}\,\mathrm{II}_k\,{}^{\parallel}\delta_c^b \qquad \mathrm{II}_{ab}{}^{k} = \frac{1}{m}\,\mathrm{Tr}\,\mathrm{II}^k\,{}^{\parallel}g_{ab}$

$m^2\varkappa^2 = (\mathrm{Tr}\,\mathrm{II})^2 = {}^{\perp}g^{kl}\,\mathrm{Tr}\,\mathrm{II}_k\,\mathrm{Tr}\,\mathrm{II}_l \qquad \mathrm{II}^2_{ab} = \frac{1}{m^2}(\mathrm{Tr}\,\mathrm{II})^2\,{}^{\parallel}g_{ab} = \varkappa^2\,{}^{\parallel}g_{ab}$

curvature

$$R_{\parallel a\parallel b\parallel c\parallel d} = \overset{\parallel}{R}_{abcd} - \varkappa^2\left({}^{\parallel}g_{ac}\,{}^{\parallel}g_{bd} - {}^{\parallel}g_{ad}\,{}^{\parallel}g_{bc}\right)$$

$$R_{\parallel a\parallel b}{}^{\perp m}{}_{\perp n} = \overset{\perp}{R}_{ab}{}^{m}{}_{n}$$

$$R_{\parallel a\parallel b}{}^{\perp m}{}_{\parallel c} = \frac{1}{m}\left(\overset{\perp}{\nabla}_a \mathrm{Tr}\,\mathrm{II}^m\,{}^{\parallel}g_{bc} - \overset{\perp}{\nabla}_b \mathrm{Tr}\,\mathrm{II}^m\,{}^{\parallel}g_{ac}\right)$$

$$R_{\parallel a\parallel b\parallel c\perp n} = \frac{1}{m}\left({}^{\parallel}g_{ca}\,\overset{\perp}{\nabla}_b \mathrm{Tr}\,\mathrm{II}_n - {}^{\parallel}g_{cb}\,\overset{\perp}{\nabla}_a \mathrm{Tr}\,\mathrm{II}_n\right)$$

contractions of curvature

$$R_{\parallel c\parallel a}{}^{\parallel c}{}_{\parallel b} = \mathrm{Ric}_{ab} - (m-1)\varkappa^2\,{}^{\parallel}g_{ab} = \mathrm{Ric}_{\parallel a\parallel b} - R_{\perp k\parallel a}{}^{\perp k}{}_{\parallel b}$$

$$R_{\parallel c\parallel a}{}^{\parallel c}{}_{\perp n} = \frac{m-1}{m}\,\overset{\perp}{\nabla}_a \mathrm{Tr}\,\mathrm{II}_n = \mathrm{Ric}_{\parallel a\perp n} - R_{\perp k\parallel a}{}^{\perp k}{}_{\perp n} = \mathrm{Ric}_{\parallel a\perp n} - R_{\perp k\perp n}{}^{\perp k}{}_{\parallel a}$$

$$R_{\parallel a\parallel b}{}^{\parallel m}{}_{\parallel m} = \mathrm{Tr}\,\overset{\parallel}{R}_{ab} = 0 \qquad R_{\parallel a\parallel b}{}^{\perp m}{}_{\perp m} = \mathrm{Tr}\,\overset{\perp}{R}_{ab} = 0 \qquad \mathrm{Tr}\,R_{ab} = 0$$

$$R_{\parallel a\parallel b}{}^{\parallel a\parallel b} = \overset{\parallel}{R} - m(m-1)\varkappa^2 = R - 2\,\mathrm{Ric}_{\perp k}{}^{\perp k} + R_{\perp k\perp l}{}^{\perp k\perp l}$$

Submanifold in Einstein space

$$Ric_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0 \qquad \Lambda = \frac{(m-1)(m-2)}{2L^2}$$

$$\Downarrow$$

$$Ric_{ab} = \frac{(m-1)}{L^2} g_{ab} \qquad R = \frac{m(m-1)}{L^2}$$

$$^{\shortparallel}Ric_{ab} = Ric_{\shortparallel a \shortparallel b} - R_{\perp k \shortparallel e}{}^{\perp k}{}_{ab} + \mathrm{Tr}\,\mathrm{II}_k\, \mathrm{II}^k_{ab} - \mathrm{II}^2_{ab}$$

$$= \frac{m-1}{L^2}\, {}^{\shortparallel}g_{ab} - R_{\perp k \shortparallel e}{}^{\perp k}{}_{\shortparallel b} + \mathrm{Tr}\,\mathrm{II}_k\, \mathrm{II}^k_{ab} - \mathrm{II}^2_{ab}$$

$$\frac{1}{n(n-1)} {}^{\shortparallel}R = \frac{1}{L^2} + \frac{1}{n(n-1)}\left((\mathrm{Tr}\,\mathrm{II})^2 - \mathrm{Tr}\,\mathrm{II}^2 \right) + \frac{1}{n(n-1)} \underbrace{\left(R_{\perp k \perp l}{}^{\perp k \perp l} - \frac{(m-n)(m-n-1)}{L^2} \right)}_{\substack{1)\ m = n+1 \\ 2)\ R_{\perp\perp\perp\perp} = \frac{1}{2L^2}\, {}^{\perp}g \wedge {}^{\perp}g}} \Rightarrow 0$$

umbilic

$$\mathrm{Tr}\,\mathrm{II}_k\, \mathrm{II}^k_{ab} - \mathrm{II}^2_{ab} = (n-1)\,\varkappa^2\, {}^{\shortparallel}g_{ab}$$

$$\frac{1}{n(n-1)}\left((\mathrm{Tr}\,\mathrm{II})^2 - \mathrm{Tr}\,\mathrm{II}^2 \right) = \varkappa^2$$

Submanifold in maximally symmetric space

$g \quad \nabla \qquad \nabla g = 0 \qquad T = 0$

$\mathrm{II}_{akb} = \mathrm{II}_{akb} \qquad \mathrm{II}^{k}_{ab} = \mathrm{II}^{k}_{ba} \qquad \mathrm{II}^{2}_{ab} = \mathrm{II}^{2}_{ba} \qquad \mathcal{H}^2 = \frac{1}{M^2}(\mathrm{Tr}\,\mathrm{II})^2$

maximally symetric space

$$R_{abcd} = \frac{1}{L^2}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) \qquad R = \frac{1}{L^2}\frac{1}{2}g \wedge g$$

$$Ric_{ab} = \frac{m-1}{L^2}\, g_{ab}$$

$$R = \frac{m(m-1)}{L^2} = \mathrm{const}$$

curvature splitting

$$^{\parallel}R_{abcd} = \frac{1}{L^2}\left({}^{\parallel}g_{ac}\,{}^{\parallel}g_{bd} - {}^{\parallel}g_{ad}\,{}^{\parallel}g_{bc}\right) + \left(\mathrm{II}^{k}_{ac}\,\mathrm{II}^{l}_{bd} - \mathrm{II}^{k}_{ad}\,\mathrm{II}^{l}_{bc}\right) {}^{+}g_{kl}$$

$$^{+}R_{ab}{}^{mn} = \left(\mathrm{II}^{m}_{ac}\,\mathrm{II}^{n}_{bd} - \mathrm{II}^{m}_{bc}\,\mathrm{II}^{n}_{ad}\right) {}^{\parallel}g^{cd}$$

$$\bar{\nabla}_a \mathrm{II}^{m}_{bn} = \bar{\nabla}_b \mathrm{II}^{m}_{an}$$

$$^{\parallel}Ric_{ab} = \frac{m-1}{L^2}\,{}^{\parallel}g_{ab} + \mathrm{Tr}\,\mathrm{II}_k\, \mathrm{II}^{k}_{ab} - \mathrm{II}^{2}_{ab}$$

$$^{+}\nabla_a \mathrm{Tr}\,\mathrm{II}^{m} = \bar{\nabla}_n \mathrm{II}_a{}^{mn}$$

$$\frac{1}{m(m-1)}\,{}^{\parallel}R = \frac{1}{L^2} + \frac{1}{m(m-1)}\left((\mathrm{Tr}\,\mathrm{II})^2 - \mathrm{Tr}\,\mathrm{II}^2\right)$$

Totally umbilic submanifold of maximally sym. space

$g \quad \nabla \qquad \nabla g = 0 \qquad T = 0$

$\mathbb{I}_{akb} = \overline{\mathbb{I}}_{akb} \qquad \mathbb{I}^{k}_{ab} = \mathbb{I}^{k}_{ba} \qquad \mathbb{I}^{?}_{ab} = \mathbb{I}^{?}_{ba}$

umbilic

$\mathbb{I}^{k}_{ab} = \frac{1}{m} \operatorname{Tr}\mathbb{I}^{k} \, {}^{\parallel}g_{ab} \qquad (\operatorname{Tr}\mathbb{I})^2 = m^2 \varkappa^2 \qquad \mathbb{I}^{2}_{ab} = \varkappa^{?} \, {}^{\parallel}g_{ab} \qquad \operatorname{Tr}\mathbb{I}^{2} = m \varkappa^2$

maximally symetric space

$R_{abcd} = \frac{1}{L^2} (g_{ac} g_{bd} - g_{ad} g_{bc})$

$Ric_{ab} = \frac{m-1}{L^2} g_{ab}$

$\mathcal{R} = \frac{m(m-1)}{L^2} = \text{const}$

curvature splitting

${}^{\parallel}\!R_{abcd} = \left(\frac{1}{L^2} + \varkappa^2\right)\left({}^{\parallel}g_{ac} \, {}^{\parallel}g_{bd} - {}^{\parallel}g_{ad} \, {}^{\parallel}g_{bc}\right)$

maximally symmetric $\qquad \frac{1}{\ell^2} = \frac{1}{L^2} + \varkappa^2 = \text{const}$

$\mathbb{R}_{ab}{}^{m}{}_{n} = 0$

$\overset{+}{\nabla}_a \operatorname{Tr}\mathbb{I}_n = 0$

${}^{\parallel}Ric_{ab} = \frac{m-1}{\ell^2} \, {}^{\parallel}g_{ab}$

${}^{\parallel}\mathcal{R} = \frac{m(m-1)}{\ell^2}$

$\frac{1}{\ell^2} = \frac{1}{L^2} + \varkappa^2$

Hypersurface embedding

$\dim N\Sigma = 1 \qquad \dim M = \dim \Sigma + 1$

normalized normal

ν normalized normal covector
n normalized normal vector
$\Big\}$ dual $\; n\cdot\nu = 1$

$^{\perp}\delta = n\nu \qquad {}^{\parallel}\delta = \delta - {}^{\perp}\delta$

metric on normal bundle

$^{\perp}g = s\,\nu\nu \qquad \nu = s\,{}^{\perp}g\cdot n \qquad n = s\,{}^{\perp}g^{-1}\cdot\nu \qquad s = \pm 1$

shortened notation

$A^{\dots\perp\dots} = A^{\dots k\dots}\nu_k \qquad A_{\dots\perp\dots} = A_{\dots k\dots}n^k \qquad A_{\cdot\perp\cdot} = s\,A^{\cdot\perp\cdot}$

in general – no metric $^{\parallel}g$ on $T\Sigma$

∇ general normal-flat covariant der. on TM

$\overset{\perp}{\nabla} n = 0 \qquad \overset{\perp}{\nabla}\nu = 0 \qquad$ i.e. $\; {}^{\perp}(\nabla_{\parallel} n) = 0 \qquad {}^{\perp}(\nabla_{\parallel}\nu) = 0$

$\Downarrow$

$\overset{\perp}{\nabla}\,{}^{\perp}g = 0 \qquad \overset{\perp}{R} = 0$

alternative definition:

we assume only projectors $^{\perp}\delta, {}^{\parallel}\delta$, not a normal ν

∇ is normal-flat cov. der., i.e. $\overset{\perp}{R} = 0$

$\Rightarrow$ there exists cov. constant normal vector n, $\overset{\perp}{\nabla} n = 0$

a choice of a scale at one point defines global n and ν

Extrinsic curvature

$$\mathrm{II}_{ab}{}^{k} = -K_{ab}\, n^k \qquad K_{ab} = -\mathrm{II}_{ab}{}^{k}\, \nu_k \qquad \mathrm{II}^2_{ab} = K_a{}^m K_{bm}$$

$$\bar{\mathrm{II}}_{ak}{}^{b} = -K_a{}^b\, \nu_k \qquad K_a{}^b = -\bar{\mathrm{II}}_{ak}{}^{b}\, n^k \qquad -\mathrm{Tr}\,\mathrm{II}_m = \vartheta\, \nu_m \quad \vartheta = K_a{}^a = -\mathrm{Tr}\,\mathrm{II}_\perp$$

$$\mathcal{H}^2 = s\left(\frac{\vartheta}{m}\right)^2$$

derivatives of normals along Σ

$$\nabla_{\parallel a}\nu_b = K_{ab} \qquad \Leftarrow \nabla_{\parallel a}\nu_b = \bar\nabla_a \nu_b - H_{ab}^k \nu_k = -\mathrm{II}_{ab}{}^k \nu_k = K_{ab}$$

$$\nabla_{\parallel a} n^b = K_a{}^b \qquad \Leftarrow \nabla_{\parallel a} n^b = \bar\nabla_c n^b + H^b_{ak} n^k = -\bar{\mathrm{II}}_{ak}{}^b n^k = K_a{}^b$$

derivatives of projectors along Σ

$$\nabla_{\parallel a}{}^{\perp}\delta^c_b = -\nabla_{\parallel a}{}^{\parallel}\delta^c_b = K_{ab} n^c + K_a{}^c \nu_b \qquad \Leftarrow \nabla_{\parallel a}{}^{\perp}\delta^c_b = \nabla_{\parallel a}(n^c \nu_b) = K_{ab} n^c + K_a{}^c \nu_b$$

$$(\nabla_{\parallel a}{}^{\perp}\delta)^{\perp}{}_{\parallel b} = K_{ab} \qquad (\nabla_{\parallel a}{}^{\perp}\delta)^{\parallel b}{}_{\perp} = K_a{}^b$$

projections of the torsion

$$T_{\parallel a \parallel b}{}^{\parallel c} = \mathbb{T}^c{}_{ab}$$

$$T_{\parallel a \parallel b}{}^{\perp} = -K_{ab} + K_{ba} \qquad \Leftarrow T_{\parallel a \parallel b}{}^{\perp} = \mathrm{II}^{\perp}_{ab} - \mathrm{II}^{\perp}_{ba}$$

curvature splitting

$$R_{\parallel a \parallel b}{}^{\parallel m}{}_{\parallel n} = {}^{\parallel}\!R_{ab}{}^m{}_n - K_a{}^m K_{bn} + K_b{}^m K_{an}$$

$$R_{\parallel a \parallel b}{}^{\perp}{}_{\perp} = -K_{ak} K_b{}^k + K_{bk} K_a{}^k$$

$$R_{\parallel a \parallel b}{}^{\perp}{}_{\parallel c} = -\left({}^{{}^{\parallel}\nabla}d_a K_b\right)_c = -{}^{\parallel}\nabla_a K_{bc} + {}^{\parallel}\nabla_b K_{ac} - \mathbb{T}^n_{ab} K_{nc}$$

$$R_{\parallel a \parallel b}{}^{\parallel c}{}_{\perp} = \left({}^{{}^{\parallel}\nabla}d_a K_b\right)^c = {}^{\parallel}\nabla_a K_b{}^c - {}^{\parallel}\nabla_b K_a{}^c + \mathbb{T}^n_{ab} K_n{}^c$$

contraction of the curvature

$$R_{\parallel c \parallel a}{}^{\parallel c}{}_{\parallel b} = {}^{\parallel}\!Ric_{ab} - \vartheta K_{ab} + K_a{}^c K_{cb}$$

$$R_{\parallel a \parallel b}{}^{\parallel c}{}_{\parallel c} = \mathrm{Tr}\,{}^{\parallel}\!R_{ab} - K_a{}^m K_{bm} + K_b{}^m K_{am}$$

$$R_{\parallel a \parallel b}{}^{\perp m}{}_{\perp m} = K_a{}^m K_{bm} - K_b{}^m K_{am}$$

$$\mathrm{Tr}\,R_{\parallel a \parallel b} = \mathrm{Tr}\,{}^{\parallel}\!R_{ab}$$

$$R_{\parallel c \parallel a}{}^{\parallel c}{}_{\perp} = {}^{\parallel}\nabla_c K_a{}^c - {}^{\parallel}\nabla_a \vartheta + K_m{}^n \mathbb{T}^m_{na}$$

Metric embedding of the hypersurface

metric on M

$$g = s\,\nu\nu + q \qquad {}^{\perp}g = s\,\nu\nu \qquad {}^{\parallel}g = q \qquad s = \pm 1 \qquad s^2 = 1$$

metric derivative

$$\nabla g = 0 \;\Rightarrow\; {}^{\parallel}\nabla q = 0 \qquad {}^{\perp}\nabla \nu = 0 \qquad {}^{\perp}R = 0 \qquad \text{general } T$$

$$\mathrm{II}_{akb} = \bar{\mathrm{II}}_{akb} \qquad K_{ab} = s\,K_{ab}$$

curvature splitting

$$R_{\parallel a \parallel b \parallel c \parallel d} = {}^{\parallel}R_{abcd} - s\,(K_{ac}K_{bd} - K_{ad}K_{bc})$$

$$R_{\parallel a \parallel b \parallel c \perp} = ({}^{\parallel}\mathrm{d}_a K_b)_c = {}^{\parallel}\nabla_a K_{bc} - {}^{\parallel}\nabla_b K_{ac} + T^m_{ab} K_{mc}$$

Levi-Civita derivative

$$T^c_{ab} = 0 \qquad K_{ab} = K_{ba} = s\,K_{ab} \qquad K^2_{ab} = K_{ac}K_{bd}\,q^{cd} = s\,\mathrm{II}^2_{ab}$$

$$k = K_a{}^a = -\mathrm{Tr}\,\mathrm{II}_\perp \qquad m^2\varkappa^2 = s\,k^2 \qquad K^2 = K^2{}_a{}^a = s\,\mathrm{Tr}\,\mathrm{II}^2$$

curvature splitting

$$R_{\parallel a \parallel b \parallel c \parallel d} = {}^{\parallel}R_{abcd} - s\,(K_{ac}K_{bd} - K_{ad}K_{bc})$$

$$R_{\parallel a \parallel b \parallel c \perp} = {}^{\parallel}\nabla_c K_{bc} - {}^{\parallel}\nabla_b K_{ac}$$

$$R_{\parallel c \parallel a}{}^{\parallel c}{}_{\parallel b} \equiv Ric_{\parallel a \parallel b} - s\,R_{\perp \parallel c \perp \parallel b} = {}^{\parallel}Ric_{ab} - s\,(k K_{ab} - K^2_{ab})$$

$$R_{\parallel c \parallel a}{}^{\parallel c}{}_{\perp} = Ric_{\perp \parallel a} = {}^{\parallel}\nabla_c K_a{}^c - {}^{\parallel}\nabla_a k$$

$$R_{\parallel a \parallel b}{}^{\parallel a \parallel b} \equiv R - 2s\,Ric_{\perp\perp} = {}^{\parallel}R - s\,(k^2 - K^2)$$

Gauss-Codazzi identity

$$R = {}^{\parallel}R + 2s\,Ric_{\perp\perp} - s\,(k^2 - K^2)$$

normal components of the Einstein tensor

$$Ric_{\perp\perp} = \frac{s}{2}(R - {}^{\parallel}R) + \frac{1}{2}(k^2 - K^2)$$

$$Ein_{\perp\perp} = Ric_{\perp\perp} - \frac{s}{2}R = -\frac{s}{2}\,{}^{\parallel}R + \frac{1}{2}(k^2 - K^2)$$

$$Ein_{\perp \parallel a} = Ric_{\perp \parallel a} = {}^{\parallel}\nabla_c K_a{}^c - {}^{\parallel}\nabla_a k$$

embedding into an Einstein space

$$Ric - \frac{1}{2}R\,g + \Lambda g = 0 \qquad \frac{1}{L^2} = \frac{2\Lambda}{(m-1)(m-2)} \qquad Ric = \frac{1}{m}R\,g = \frac{m-1}{L^2}g \qquad R = \frac{m(m-1)}{L^2}$$

$$\Rightarrow\; R - 2s\,Ric_{\perp\perp} = \frac{m(m-1)}{L^2} - 2s^2\frac{m-1}{L^2} = \frac{n(n-1)}{L^2} \qquad n = m-1$$

$$\text{G-C} \;\Rightarrow\; \frac{1}{\ell^2} \equiv \frac{1}{n(n-1)}\,{}^{\parallel}R = \frac{1}{L^2} + \frac{s}{n(n-1)}(k^2 - K^2) \qquad \text{(not necessarily constant)}$$

embedding into a maximally symmetric space

$$R_{abcd} = \frac{1}{L^2}(g_{ac}g_{bd} - g_{ad}g_{bc}) \qquad Ric_{ab} = \frac{m-1}{L^2} g_{ab} \qquad R = \frac{m(m-1)}{L^2} = \text{const.}$$

$$\Downarrow$$

$${}^{\parallel}R_{abcd} = \frac{1}{L^2}(q_{ac}q_{bd} - q_{ad}q_{bc}) + s(K_{ac}K_{bd} - K_{ad}K_{bc})$$

$${}^{\parallel}Ric_{ab} = \frac{m-1}{L^2} q_{ab} + s(k K_{ab} - K^2_{ab})$$

$$\frac{1}{\ell^2} \equiv \frac{1}{m(m-1)}\, {}^{\parallel}R = \frac{1}{L^2} + \frac{s}{m(m-1)}(k^2 - K^2)$$

$${}^{\parallel}\nabla_a K_{bc} = {}^{\parallel}\nabla_b K_{ac} \qquad {}^{\parallel}\nabla_c K_a{}^c = {}^{\parallel}\nabla_a k$$

umbilic embedding

$$K_{ab} = \frac{1}{m} k\, q_{ab} \qquad K^2_{ab} = \frac{1}{m^2} k^2 q_{ab} \qquad K^2 = \frac{1}{m} k^2 \qquad m^2 \varkappa^2 = s k^2$$

$$\Downarrow$$

$$R_{\parallel a \parallel b \parallel c \parallel d} = {}^{\parallel}R_{abcd} - \varkappa^2(q_{ac}q_{bd} - q_{ad}q_{bc})$$

$$Ric_{\parallel a \parallel b} - s R_{\perp \parallel a \perp \parallel b} = {}^{\parallel}Ric_{ab} - (m-1)\varkappa^2 q_{ab}$$

$$R - 2s\, Ric_{\perp\perp} \equiv -2s\, Ein_{\perp\perp} = {}^{\parallel}R - m(m-1)\varkappa^2$$

$$R_{\parallel a \parallel b \parallel c \perp} = {}^{\parallel}\nabla_a k\, q_{bc} - {}^{\parallel}\nabla_b k\, q_{ca}$$

$$Ric_{\perp \parallel a} = -\frac{m-1}{m}\, {}^{\parallel}\nabla_a k$$

$$\Downarrow \quad \frac{1}{\ell^2} \equiv \frac{1}{m(m-1)}\, {}^{\parallel}R = \frac{1}{m(m-1)}(R - 2s\, Ric_{\perp\perp}) + \varkappa^2$$

$$= -\frac{2s}{m(m-1)} Ein_{\perp\perp} + \varkappa^2$$

umbilic embedding into maximally symmetric space

$${}^{\parallel}R_{abcd} = \left(\frac{1}{L^2} + \varkappa^2\right)(q_{ac}q_{bd} - q_{ad}q_{bc}) \qquad \frac{1}{\ell^2} = \frac{1}{L^2} + \varkappa^2$$

$${}^{\parallel}Ric_{ab} = \frac{m-1}{\ell^2} q_{ab}$$

$${}^{\parallel}R = \frac{m(m-1)}{\ell^2}$$

$${}^{\parallel}\nabla_a k = 0$$

$$\frac{1}{\ell^2} = \frac{1}{L^2} + \varkappa^2 = \text{const.} \qquad \varkappa^2 = s\left(\frac{k}{m}\right)^2$$

Surface embedding into 3D max. sym. space

$m = 3 \qquad n = 2 \qquad s = + \qquad \operatorname{sign} g = (++)$

$K = k_+ \, e^+ e^+ + k_- \, e^- e^- \qquad g = e^+ e^+ + e^- e^-$

$k = k_+ + k_- \qquad K^2 = k_+^2 + k_-^2 \qquad k^2 - K^2 = 2 k_+ k_-$

embedding into $\mathbb{E}^3 \qquad R = 0 = \frac{1}{L^2}$

Gauss-Codazzi $\Rightarrow$

$$\underbrace{\frac{1}{\ell^2} \equiv \frac{1}{2} \mathbb{R}}_{\text{inner curvature}} = \underbrace{k_+ k_-}_{\text{extrinsic curvature}}$$

Theoreme Egregium (Gauss)

embedding into max. sym. space = sphere/eucl./hyperbolic

$$\frac{1}{\ell^2} \equiv \frac{1}{2} \mathbb{R} = \frac{1}{L^2} + k_+ k_-$$

$\frac{1}{L^2}$:

- > 0 sphere S^3
- $= 0$ euclidian sp. E^3
- < 0 hyperbolic sp. $\mathbb{H}^3$

$R = \frac{6}{L^2}$